The Feynman-Vernon Influence Functional Approach in QED and Lamb Shift

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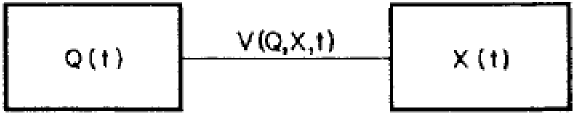
The XXIII International Workshop
High Energy Physics and Quantum Field Theory
June 26–July 3, 2017
Yaroslavl, Russia

The Feynman-Vernon Influence Functional approach

R.P. Feynman, F.L. Vernon, Jr. **The Theory of a General Quantum System Interacting with a Linear Dissipative System**, *Annals of Physics* **24**, 118–173.



General quantum systems Q and X coupled by a potential $V(Q, X, t)$.

«. . . It is shown that the effect of the external systems in such a formalism [paths integral formalism] can always be included in a general class of functionals (influence functionals) of the coordinates of the system only . . . »

QED Lagrangian

$$\mathcal{L}(x) = \overline{\psi}(x)(\imath\gamma^{\mu}\partial_{\mu} - m)\psi(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - ej^{\mu}(x)A_{\mu}(x) \tag{1}$$

where

$$F_{\mu\nu}(x) = \partial_{\mu}A_{\mu}(x) - \partial_{\nu}A_{\mu}(x) \tag{2}$$

$$j^{\mu}(x) = \overline{\psi}(x)\gamma^{\mu}\psi(x) \tag{3}$$

$$\hat{\psi}(\mathbf{x},t) = \sum_{\mathbf{p},\sigma=1,2} \frac{1}{\sqrt{2V\omega_{\mathbf{p}}^{(f)}}} \left(\hat{b}_{\mathbf{p}\sigma} u_\sigma(p) e^{\imath px} + \hat{c}^\dagger_{\mathbf{p}\sigma} u_\sigma(-p) e^{-\imath px} \right), \tag{4}$$

$$\hat{\overline{\psi}}(\mathbf{x},t) = \sum_{\mathbf{p},\sigma=1,2} \frac{1}{\sqrt{2V\omega_{\mathbf{p}}^{(f)}}} \left(\hat{b}^\dagger_{\mathbf{p}\sigma} \overline{u}_\sigma(p) e^{-\imath px} + \hat{c}_{\mathbf{p}\sigma} \overline{u}_\sigma(-p) e^{\imath px} \right), \tag{5}$$

$$\hat{j}_\mu(\mathbf{x},t) = \hat{\overline{\psi}}(\mathbf{x},t)\gamma_\mu \hat{\psi}(\mathbf{x},t) \tag{6}$$

$$\hat{A}^\mu(\mathbf{x},t) = \sum_{\mathbf{k},\lambda=1,2} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}^{(b)}}} \varepsilon^\mu_\lambda \left(\hat{a}_{\mathbf{k}\lambda} e^{\imath kx} + \hat{a}^\dagger_{\mathbf{k}\lambda} e^{-\imath kx} \right) \tag{7}$$

Second quantization

$$\hat{H}_{full} = \sum_{\mathbf{p},\sigma=1,2} \omega_{\mathbf{p}}^{(f)} \left(\hat{b}^{\dagger}_{\mathbf{p}\sigma} \hat{b}_{\mathbf{p}\sigma} + \hat{c}^{\dagger}_{\mathbf{p}\sigma} \hat{c}_{\mathbf{p}\sigma} \right) + \sum_{\mathbf{k},\lambda=1,2} \omega_{\mathbf{k}}^{(b)} \hat{a}^{\dagger}_{\mathbf{k}\lambda} \hat{a}_{\mathbf{k}\lambda} + \\ + e \sum_{\mathbf{k},\lambda=1,2} \frac{\imath}{\sqrt{2\omega_{\mathbf{k}}^{(b)} V}} \left(\varepsilon^{\mu}_{\lambda} \hat{j}^{+}_{\mu}(\mathbf{k}, t) \hat{a}_{\mathbf{k}\lambda} + \varepsilon^{*\mu}_{\lambda} \hat{j}^{-}_{\mu}(\mathbf{k}, t) \hat{a}^{\dagger}_{\mathbf{k}\lambda} \right) \quad (8)$$

where

$$\hat{j}^{+}_{\mu}(\mathbf{k}, t) = \int \hat{j}_{\mu}(\mathbf{x}, t) e^{\imath \mathbf{k}\mathbf{x}} d\mathbf{x}, \qquad \hat{j}^{-}_{\mu}(\mathbf{k}, t) = \int \hat{j}_{\mu}(\mathbf{x}, t) e^{-\imath \mathbf{k}\mathbf{x}} d\mathbf{x} \quad (9)$$

Evolution equation for statistical operator $\hat{\rho}(t_f)$

$$\hat{\rho}(t_f) = \hat{U}(t_f, t_{in})\hat{\rho}(t_{in})\hat{U}^{\dagger}(t_f, t_{in}) \tag{10}$$

where $\hat{\rho}(t_{in})$ is statistical operator, describing initial state at moment t_{in}, $\hat{U}(t_f, t_{in})$ — evolution operator.

$$\hat{U}(t_f, t_{in}) = \hat{T} \exp[-\frac{\imath}{\hbar}\int_{t_{in}}^{t_f} \hat{H}_{full}(\tau) d\tau]. \tag{11}$$

where $\hat{H}_{full}$:

$$\hat{H}_{full} = \hat{H}_{sys} + \hat{H}_{field} + \hat{H}_{int} \tag{12}$$

Coherent states for electromagetic field

$$\hat{a}_{\mathbf{k}\lambda}|\alpha_{\mathbf{k}\lambda}\rangle = \alpha_{\mathbf{k}\lambda}|\alpha_{\mathbf{k}\lambda}\rangle, \qquad \langle\alpha_{\mathbf{k}\lambda}|\hat{a}^{\dagger}_{\mathbf{k}\lambda} = \langle\alpha_{\mathbf{k}\lambda}|\alpha^{*}_{\mathbf{k}\lambda}, \tag{13}$$

where $\alpha_{\mathbf{k}\lambda}$ — complex value, which describe states $\mathbf{k}$ mode of quantum electromagnetic field. These states ($|\alpha\rangle$) are non-orthogonal:

$$\langle\alpha'_{\mathbf{k}'\lambda'}|\alpha_{\mathbf{k}\lambda}\rangle = \delta_{\mathbf{k}'\mathbf{k}}\delta_{\lambda'\lambda}\exp\left\{-\frac{1}{2}\left(|\alpha'_{\mathbf{k}'\lambda'}|^2 + |\alpha_{\mathbf{k}\lambda}|^2 - 2\alpha'^{*}_{\mathbf{k}'\lambda'}\alpha_{\mathbf{k}\lambda}\right)\right\}. \tag{14}$$

There is resolution of the identity operator:

$$\int |\alpha_{\mathbf{k}\lambda}\rangle\langle\alpha_{\mathbf{k}\lambda}|\frac{d^2\alpha_{\mathbf{k}\lambda}}{\pi} = \hat{1}. \tag{15}$$

Grassman states for Dirac field

$$\hat{b}_{\mathbf{p},\sigma}|\theta_{\mathbf{p},\sigma}\rangle = \theta_{\mathbf{p},\sigma}|\theta_{\mathbf{p},\sigma}\rangle, \qquad \langle\overline{\theta}_{\mathbf{p},\sigma}|\hat{b}^{\dagger}_{\mathbf{p},\sigma} = \langle\overline{\theta}_{\mathbf{p},\sigma}|\overline{\theta}_{\mathbf{p},\sigma}, \tag{16}$$

where $\theta_{\mathbf{p},\sigma}$ — grassman variable. These states $(|\theta\rangle)$ are non-orthogonal:

$$\langle\overline{\theta}'_{\mathbf{p}'\sigma'}|\theta_{\mathbf{p}\sigma}\rangle = \delta_{\mathbf{p}'\mathbf{p}}\delta_{\sigma'\sigma}\exp\left\{-\frac{1}{2}\left(\overline{\theta}'_{\mathbf{p}'\sigma'}\theta'_{\mathbf{p}'\sigma'} + \overline{\theta}_{\mathbf{p}\sigma}\theta_{\mathbf{p}\sigma} - 2\overline{\theta}'_{\mathbf{p}'\sigma'}\theta_{\mathbf{p}\sigma}\right)\right\}. \tag{17}$$

There is resolution of the identity operator:

$$\int |\theta_{\mathbf{p}\sigma}\rangle\langle\overline{\theta}_{\mathbf{p}\sigma}|\frac{d^2\theta_{\mathbf{p}\sigma}}{\pi} = \hat{1}. \tag{18}$$

Grassman variables properties

For two grassman variables θ and η

$$\theta\eta + \eta\theta = 0 \qquad \text{or} \qquad \theta\eta = -\eta\theta \qquad \text{so} \qquad \theta^2 = 0 \tag{19}$$

Then

$$\int d\theta f(\theta) = \int d\theta (A + B\theta) = B. \tag{20}$$

and

$$\int d\theta^* d\theta e^{-\theta^* b\theta} = b \tag{21}$$

and

$$\left(\prod_i \int d\theta_i^* d\theta\right) \theta_k \theta_l^* e^{-\theta_i^* B_{ij}\theta_j} = \frac{\det B}{B_{kl}} \tag{22}$$

Evolution equation for density matrix in holomorphic representation $|\theta_{\mathbf{p}\sigma}, \alpha_{\mathbf{k}\lambda}\rangle = |\theta_{\mathbf{p}\sigma}\rangle \otimes |\alpha_{\mathbf{k}\lambda}\rangle$

The density matrix:

$$\rho(\alpha_f^*, \overline{\theta}_f, \alpha_f', \theta_f'; t_f) = \langle \overline{\theta}_f, \alpha_f | \hat{\rho}(t_f) | \theta_f', \alpha_f' \rangle \tag{23}$$

The kernel of evolution operator:

$$U(\alpha_f^*, \overline{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) = \langle \overline{\theta}_f, \alpha_f | \hat{U}(t_f, t_{in}) | \theta_{in}, \alpha_{in} \rangle \tag{24}$$

The evolution equation:

$$\rho(\alpha_f^*, \overline{\theta}_f, \alpha_f', \theta_f'; t_f) = \int \frac{d^2\alpha_{in}'}{\pi} \frac{d^2\theta_{in}'}{\pi} \frac{d^2\alpha_{in}}{\pi} \frac{d^2\theta_{in}}{\pi} \times$$

$$\times U(\alpha_f^*, \overline{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) \rho(\alpha_{in}^*, \overline{\theta}_{in}, \alpha_{in}', \theta_{in}'; t_{in}) U^*(\alpha_f', \theta_f', t_f | \alpha_{in}'^*, \overline{\theta}_{in}'; t_{in}) \tag{25}$$

The kernel of evolution operator

$$U(\alpha_f^*, \overline{\theta}_f, t_f | \alpha_f, \theta_f, t_{in}) = \int \mathfrak{D}\alpha^*(\tau)\mathfrak{D}\alpha(\tau)\mathfrak{D}\overline{\theta}(\tau)\mathfrak{D}\theta(\tau) \times$$
$$\times \exp\left\{\imath S_{full}\left[\alpha^*(\tau), \alpha(\tau), \overline{\theta}(\tau), \theta(\tau)\right]\right\}, \qquad (26)$$

where action

$$S_{full}\left[\alpha^*(\tau), \alpha(\tau), \overline{\theta}(\tau), \theta(\tau)\right] =$$
$$= S_f\left[\overline{\theta}(\tau), \theta(\tau)\right] + S_b\left[\alpha^*(\tau), \alpha(\tau)\right] + S_{int}\left[\alpha^*(\tau), \alpha(\tau), \overline{\theta}(\tau), \theta(\tau)\right]. \quad (27)$$

Action of fermionic field:

$$S_f\left[\overline{\theta}(\tau),\theta(\tau)\right]=\int_{t_{in}}^{t_f}\left(\frac{\dot{\overline{\theta}}(\tau)\theta(\tau)-\overline{\theta}(\tau)\dot{\theta}(\tau)}{2\imath}-\omega^{(f)}\overline{\theta}(\tau)\theta(\tau)\right)d\tau \tag{28}$$

Action of bosonic field:

$$S_b\left[\alpha^*(\tau),\alpha(\tau)\right]=\int_{t_{in}}^{t_f}\left(\frac{\dot{\alpha}^*(\tau)\alpha(\tau)-\alpha^*(\tau)\dot{\alpha}(\tau)}{2\imath}-\omega^{(b)}\alpha^*(\tau)\alpha(\tau)\right)d\tau \tag{29}$$

Action of interaction part:

$$S_{int}\left[\alpha^*(\tau),\alpha(\tau),\overline{\theta}(\tau),\theta(\tau)\right]=\int_{t_{in}}^{t_f}ej^{\mu}(\overline{\theta}(\tau),\theta(\tau))(\varepsilon_{\mu}^{*}\alpha^*(\tau)+\varepsilon_{\mu}\alpha(\tau))]d\tau; \tag{30}$$

Evolution of density matrix in paths integral formulation

We have

$$\rho(\alpha^*_f, \overline{\theta}_f, \alpha'_f, \theta'_f; t_f) = \int \frac{d^2\alpha'_{in}}{\pi}\frac{d^2\theta'_{in}}{\pi}\frac{d^2\alpha_{in}}{\pi}\frac{d^2\theta_{in}}{\pi}\rho(\alpha^*_{in}, \overline{\theta}_{in}, \alpha'_{in}, \theta'_{in}; t_{in})\times$$
$$\times \mathfrak{D}\alpha^*(\tau)\mathfrak{D}\alpha(\tau)\mathfrak{D}\overline{\theta}(\tau)\mathfrak{D}\theta(\tau)\mathfrak{D}\alpha'^*(\tau)\mathfrak{D}\alpha'(\tau)\mathfrak{D}\overline{\theta}'(\tau)\mathfrak{D}\theta'(\tau)\times$$
$$\times \exp\left\{\imath\left(S_{full}\left[\alpha^*(\tau), \alpha(\tau), \overline{\theta}(\tau), \theta(\tau)\right] - S_{full}\left[\alpha'^*(\tau), \alpha'(\tau), \overline{\theta}'(\tau), \theta'(\tau)\right]\right)\right\}, \qquad (31)$$

Fermionic density matrix and influence functional

$$\rho(\overline{\theta}_f,\theta'_f;t_f) = Sp_{\alpha_f=\alpha'_f}\rho(\alpha^*_f,\overline{\theta}_f,\theta'_f,\alpha'_f;t_f) = \int d\theta'_f d\theta_f \mathfrak{D}\overline{\theta}(\tau)\mathfrak{D}\theta(\tau)\mathfrak{D}\overline{\theta}'(\tau)\mathfrak{D}\theta'(\tau)d\theta'_{in}d\theta_{in}\times$$
$$\times\exp\left\{\imath\left(S_f[\overline{\theta}(\tau),\theta(\tau)] - S_f[\overline{\theta}'(\tau),\theta'(\tau)]\right)\right\}F[\theta(\tau),\theta'(\tau)] \quad (32)$$

where $F[\overline{\theta}(\tau),\theta'(\tau)]$ is influence functional of electromagnetic field on fermionic subsystems.

$$F[\overline{\theta}(\tau),\theta'(\tau)] = Sp_{\alpha_f=\alpha'_f}\int \mathfrak{D}\alpha^*(\tau)\mathfrak{D}\alpha(\tau)\mathfrak{D}\alpha'^*(\tau)\mathfrak{D}\alpha'(\tau)\frac{d^2\alpha_{in}}{\pi}\frac{d^2\alpha'_{in}}{\pi}\rho(\alpha^*_{in},\overline{\theta}_{in},\alpha'_{in},\theta'_{in};t_{in})\times$$
$$\times\exp\left\{\imath\left(S_b\left[\alpha^*(\tau),\alpha(\tau)\right] + S_{int}\left[\alpha^*(\tau),\alpha(\tau),\overline{\theta}(\tau),\theta(\tau)\right] - S_b\left[\alpha'^*(\tau),\alpha'(\tau)\right] - S_{int}\left[\alpha'^*(\tau),\alpha'(\tau),\overline{\theta}'(\tau),\theta'(\tau)\right]\right)\right\} \quad (33)$$

In many cases, we can choose at initial moment t_{in}

$$\rho(\alpha^*_{in},\overline{\theta}_{in},\alpha'_{in},\theta'_{in};t_{in}) = \rho_f(\overline{\theta}_{in},\theta'_{in};t_{in})\times\rho_b(\alpha^*_{in},\alpha'_{in};t_{in}) \quad (34)$$

Influence functional of electromagnetic field

$$F[\overline{\theta}(\tau),\theta'(\tau)] = \int Sp_{\alpha_f=\alpha'_f} \frac{d^2\alpha_{in}}{\pi}\frac{d^2\alpha'_{in}}{\pi}\times$$
$$\times U_{infl}(\alpha_f^*,\overline{\theta}_f,t_f|\alpha_{in},\theta_{in},t_{in})\rho(\alpha_{in}^*,\overline{\theta}_{in},\alpha'_{in},\theta'_{in};t_{in})U^*_{infl}(\alpha'_f,\theta'_f,t_f|\alpha'^*_{in},\overline{\theta}'_{in};t_{in}) \qquad (35)$$

where $U_{infl}(\alpha_f^*,\overline{\theta}_f,t_f|\alpha_{in},\theta_{in},t_{in})$ is electromagnetic field transition amplitude from initial state $|\alpha_{in}\rangle$ to final state $|\alpha_f^*\rangle$ inducing by external source j:

$$U_{infl}(\alpha_f^*,\overline{\theta}_f,t_f|\alpha_{in},\theta_{in},t_{in}) = \int \mathfrak{D}\alpha^*(\tau)\mathfrak{D}\alpha(\tau)\exp\{\imath S_{infl}[\alpha^*(\tau),\alpha(\tau),x(\tau)]\} \qquad (36)$$

where $S_{infl}\left[\alpha^*(\tau),\alpha(\tau),\overline{\theta}(\tau),\theta(\tau)\right] = S_b\left[\alpha^*(\tau),\alpha(\tau)\right] + S_{int}\left[\alpha^*(\tau),\alpha(\tau),\overline{\theta}(\tau),\theta(\tau)\right]$. In general, influence functional (26) describes the influence (action) of electromagnetic field on fermionic field.

Functional integration over electromagnetic field paths

$$U_{infl}(\alpha_f^*, \overline{\theta}_f, t_f|\alpha_{in}, \theta_{in}, t_{in}) = \exp\left\{ e^{-\imath\omega(t_f - t_{in})}\alpha_f^*\alpha_{in} - e^2 \int_{t_{in}}^{t_f}\int_{t_{in}}^{\tau} \varepsilon^{\mu} j_{\mu}^{+}(\tau)\varepsilon^{*\nu} j_{\nu}^{-}(\tau')e^{\imath\omega(\tau-\tau')}d\tau d\tau' - \right.$$
$$\left. - \imath\alpha_{in}e\int_{t_{in}}^{t_f}\varepsilon^{\mu} j_{\mu}^{+}(\tau)e^{-\imath\omega(\tau - t_{in})}d\tau - \imath\alpha_f^* e\int_{t_{in}}^{t_f}\varepsilon^{*\mu} j_{\mu}^{-}(\tau)e^{-\imath\omega(t_f-\tau)}d\tau \right\} \tag{37}$$

For multimode field and two polarizations without interaction between modes

$$U_{infl} = \prod_{\mathbf{k},\lambda} U_{infl}^{(\mathbf{k},\lambda)} \tag{38}$$

$$
\begin{aligned}
& U_{infl}(\alpha_f^*, \overline{\theta}_f, t_f | \alpha_{in}, \theta_{in}, t_{in}) = \\
&= \prod_{\mathbf{k},\lambda=1,2} \exp\left\{ e^{-\imath\omega_{\mathbf{k}}(t_f - t_{in})} \alpha_{\mathbf{k}\lambda}^{(f)*} \alpha_{\mathbf{k}\lambda}^{(in)} - \frac{e^2}{2\omega_{\mathbf{k}} V} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \varepsilon_\lambda^\mu j_\mu^+(\mathbf{k},\tau) \varepsilon_\lambda^{*\nu} j_\nu^-(\mathbf{k},\tau') e^{\imath\omega_{\mathbf{k}}(\tau-\tau')} d\tau d\tau' - \right. \\
& \left. - \imath\alpha_{\mathbf{k}\lambda}^{(in)} \frac{e}{\sqrt{2\omega_{\mathbf{k}} V}} \int_{t_{in}}^{t_f} \varepsilon_\lambda^\mu j_\mu^+(\mathbf{k},\tau) e^{-\imath\omega_{\mathbf{k}}(\tau - t_{in})} d\tau - \imath\alpha_{\mathbf{k}\lambda}^{(f)*} \frac{e}{\sqrt{2\omega_{\mathbf{k}} V}} \int_{t_{in}}^{t_f} \varepsilon_\lambda^{*\mu} j_\mu^-(\mathbf{k},\tau) e^{-\imath\omega_{\mathbf{k}}(t_f - \tau)} d\tau \right\} = \\
&= \exp\left\{ \sum_{\mathbf{k},\lambda} \left(e^{-\imath\omega_{\mathbf{k}}(t_f - t_{in})} \alpha_f^* \alpha_{in} - \frac{e^2}{2\omega_{\mathbf{k}} V} \int_{t_{in}}^{t_f} \int_{t_{in}}^{\tau} \varepsilon_\lambda^\mu j_\mu^+(\mathbf{k},\tau) \varepsilon_\lambda^{*\nu} j_\nu^-(\mathbf{k},\tau') e^{\imath\omega_{\mathbf{k}}(\tau-\tau')} d\tau d\tau' - \right.\right. \\
& \left.\left. - \imath\alpha_{in} \frac{e}{\sqrt{2\omega_{\mathbf{k}} V}} \int_{t_{in}}^{t_f} \varepsilon_\lambda^\mu j_\mu^+(\mathbf{k},\tau) e^{-\imath\omega_{\mathbf{k}}(\tau - t_{in})} d\tau - \imath\alpha_f^* \frac{e}{\sqrt{2\omega_{\mathbf{k}} V}} \int_{t_{in}}^{t_f} \varepsilon_\lambda^{*\mu} j_\mu^-(\mathbf{k},\tau) e^{-\imath\omega_{\mathbf{k}}(t_f - \tau)} d\tau \right) \right\} \qquad (39)
\end{aligned}
$$

Vacuum influence functional

For the case when initial and final states of electromagnetic field are vacuum:

$$\phi_{in}(\alpha_{in}) = \langle \alpha_{in}|0\rangle = \exp\left\{-\frac{1}{2}|\alpha_{in}|^2\right\}, \qquad \phi_f^*(\alpha_f) = \langle 0|\alpha_f\rangle = \exp\left\{-\frac{1}{2}|\alpha_f|^2\right\}. \quad (40)$$

We define influence functional of electromagnetic vacuum

$$F_{\langle vac|vac\rangle}[\overline{\theta}(\tau), \theta'(\tau)] = \int \frac{d^2\alpha_f}{\pi}\frac{d^2\alpha'_f}{\pi}\frac{d^2\alpha_{in}}{\pi}\frac{d^2\alpha'_{in}}{\pi}\rho_f(\overline{\theta}_{in}, \theta'_{in}; t_{in})\times$$
$$\times\phi_f^*(\alpha_f)U_{infl}(\alpha_f^*, \overline{\theta}_f, t_f|\alpha_{in}, \theta_{in}, t_{in})\phi_{in}(\alpha_{in})\phi_{in}^*(\alpha'_{in})U_{infl}^*(\alpha'_f, \theta'_f, t_f|\alpha'^*_{in}, \overline{\theta}'_{in}; t_{in})\phi_f^*(\alpha'_f) =$$
$$= \exp\left\{-\sum_{\mathbf{k},\lambda}\frac{e^2}{2\omega_{\mathbf{k}}V}\int_{t_{in}}^{t_f}\int_{t_{in}}^{\tau}\left(\varepsilon^{\mu}_{\lambda}j^{+}_{\mu}(\mathbf{k},\tau)\varepsilon^{*\nu}_{\lambda}j^{-}_{\nu}(\mathbf{k},\tau')e^{\imath\omega_{\mathbf{k}}(\tau-\tau')}d\tau d\tau' + \varepsilon^{\mu}_{\lambda}j'^{+}_{\mu}(\mathbf{k},\tau)\varepsilon^{*\nu}_{\lambda}j'^{-}_{\nu}(\mathbf{k},\tau')e^{-\imath\omega_{\mathbf{k}}(\tau-\tau')}d\tau d\tau'\right)\right\}$$

(41)

From sum over $\mathbf{k}$ to integral: $\sum\limits_{\mathbf{k}} \to \frac{V}{(2\pi)^3}\int d\mathbf{k}$

$$-\sum_{\mathbf{k}} \frac{e^2}{2\omega_{\mathbf{k}} V} \int\limits_{t_{in}}^{t_f} \int\limits_{t_{in}}^{\tau} \left[\sum_{\lambda} \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{nu}\right] j_{\mu}^{+}(\mathbf{k},\tau) j_{\nu}^{-}(\mathbf{k},\tau') e^{\imath\omega_{\mathbf{k}}(\tau-\tau')} d\tau d\tau' =$$

$$= -\frac{e^2}{(2\pi)^3} \int\limits_{t_{in}}^{t_f} \int\limits_{t_{in}}^{\tau} \int \frac{1}{2\omega_{\mathbf{k}}} \left[\sum_{\lambda} \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{*\nu}\right] j_{\mu}^{+}(\mathbf{k},\tau) j_{\nu}^{-}(\mathbf{k},\tau') e^{\imath\omega(\tau-\tau')} d\mathbf{x} d\mathbf{x}' d\mathbf{k} d\tau d\tau' =$$

$$= -\frac{e^2}{(2\pi)^3} \int\limits_{t_{in}}^{t_f} \int\limits_{t_{in}}^{\tau} \int \frac{1}{2\omega_{\mathbf{k}}} \left[\sum_{\lambda} \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{*\nu}\right] j_{\mu}(\mathbf{x},\tau) j_{\nu}(\mathbf{x}',\tau') e^{-\imath\mathbf{k}(\mathbf{x}-\mathbf{x}')} e^{\imath\omega(\tau-\tau')} d\mathbf{x} d\mathbf{x}' d\mathbf{k} d\tau d\tau' =$$

$$= -\frac{e^2}{4\pi\imath} \int\limits_{t_{in}}^{t_f} \int\limits_{t_{in}}^{\tau} \int \underbrace{\left[\frac{1}{(2\pi)^3} \int \frac{2\pi\imath d\mathbf{k}}{\omega_{\mathbf{k}}} \left(\sum_{\lambda} \varepsilon_{\lambda}^{\mu} \varepsilon_{\lambda}^{*\nu}\right) e^{-\imath\mathbf{k}(\mathbf{x}-\mathbf{x}')} e^{\imath\omega(\tau-\tau')} d\mathbf{k}\right]}_{D^{\mu\nu}(\mathbf{x}-\mathbf{x}',\tau-\tau')} j_{\mu}(\mathbf{x},\tau) j_{\nu}(\mathbf{x}',\tau') d\mathbf{x} d\mathbf{x}' d\tau d\tau'$$

where $D^{\mu\nu}(\mathbf{x}-\mathbf{x}',\tau-\tau')$ is photon propagator [1].

[1] V.B. Berestetskii, E.M. Lifshitz, L.P. Pitaevskii **Quantum Electrodynamics**

$$F_{\langle vac|vac\rangle}[\overline{\theta}(\tau),\theta'(\tau)] = \exp\left\{-\frac{e^2}{4\pi\imath}\int\limits_{t_{in}}^{t_f}\int\limits_{t_{in}}^{\tau}\int j_\mu(\mathbf{x},\tau)D^{\mu\nu}(\mathbf{x}-\mathbf{x}',\tau-\tau')j_\nu(\mathbf{x}',\tau')d\mathbf{x}d\mathbf{x}'d\tau d\tau' - \right.$$
$$\left. -\frac{e^2}{4\pi\imath}\int\limits_{t_{in}}^{t_f}\int\limits_{t_{in}}^{\tau}\int j'_\mu(\mathbf{x},\tau)D^{\mu\nu}(\mathbf{x}-\mathbf{x}',\tau-\tau')j'_\nu(\mathbf{x}',\tau')d\mathbf{x}d\mathbf{x}'d\tau d\tau'\right\} \quad (42)$$

For $t_f \to \infty$, $t_{in} \to -\infty$ we have relativistic invariant influence functional of electromagnetic vacuum:

$$F_{\langle vac|vac\rangle}[\overline{\theta}(\tau),\theta'(\tau)] = \exp\left\{-\frac{e^2}{4\pi\imath}\int\int\left(j_\mu(x)D^{\mu\nu}(x-x')j_\nu(x') + j'_\mu(x)D^{\mu\nu}(x-x')j'_\nu(x')\right)d^4x d^4x'\right\} \quad (43)$$

So we have effective Lagrangian

$$\mathcal{L} = \overline{\psi}(x)(\imath\gamma_\mu\partial^\mu - m)\psi(x) + \frac{e^2}{4\pi} j_\mu(x) \int D^{\mu\nu}(x - x')j_\nu(x')dx' \tag{44}$$

The Euler–Lagrange equation for this model is

$$(\imath\gamma_\mu\partial^\mu - m)\psi(x) + \frac{e^2}{4\pi}\left[\int D^{\mu\nu}(x - x')j_\nu(x')dx'\right]\gamma_\mu\psi(x) = 0 \tag{45}$$

where

$$D^{\mu\nu}(x - x') = \frac{1}{(2\pi)^3}\int \frac{2\pi\imath d\mathbf{k}}{\omega_\mathbf{k}}\left(\sum_\lambda \varepsilon^\mu_\lambda \varepsilon^{*\nu}_\lambda\right) e^{-\imath\mathbf{k}(\mathbf{x}-\mathbf{x}')}e^{\imath\omega(\tau-\tau')} \tag{46}$$

and

$$j_\nu(x') = \overline{\psi}(x')\gamma_\nu\psi(x') \tag{47}$$

We note that obtaibed equation is non-linear. We present equation (45) in two forms:

$$(\imath\gamma_{\mu}\partial^{\mu} - \mathcal{M})\,\psi(x) = 0 \tag{48}$$

where

$$\mathcal{M} = m - \frac{e^2}{4\pi}\left[\int D^{\mu\nu}(x - x')j_{\nu}(x')dx'\right]\gamma_{\mu} \tag{49}$$

Thanks for your attention!